Sufficient LMI conditions for reduced-order multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ control of LTI systems

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Abstract

This paper presents a novel projection lemma based linear matrix inequality (LMI) framework to design reduced-order multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ controllers for linear time-invariant systems. This framework relies on a set of full-order $\mathcal{H}_2/\mathcal{H}_\infty$ controllers, which are used as parameters in sufficient LMIs for the reduced-order controller design. Continuous-time and discrete-time controller designs are treated in a unified fashion. It is theoretically and numerically demonstrated that the approach allows the computation of reduced-order controllers that are potentially less conservative than full-order designs resulting from well-known LMI approaches. Various comparisons with existing reduced-order controller design approaches illustrate the potential of the proposed framework of sufficient LMIs.

Keywords: linear time-invariant systems, $\mathcal{H}_2/\mathcal{H}_\infty$ performance, reduced-order control, multi-objective control, linear matrix inequalities

1. Introduction

The general objective of a controller is to stabilize a given system while achieving one or more closed-loop performance specifications. Depending on the application, different types and quantities of specifications are required to arrive at a suitable closed-loop behaviour. In addition to closed-loop stability and performance, the controller structure is an important aspect as well. Related to the latter, the design of reduced-order controllers has been extensively studied since the 1960s. Especially for complicated systems, design procedures for reduced-order controllers are a fundamental necessity, since such controllers provide high reliability and low implementation cost.

Based on Lyapunov theory and the notion of dissipativity, convex reformulations of many analysis and control problems have been derived during the last decades [1–5]. The corresponding feasibility and optimization problems are formulated in terms of linear matrix inequalities (LMIs) and are thus efficiently solved by interior-point methods, which guarantee convergence to a global optimum within polynomial time [6]. The $\mathcal{H}_2$ and the $\mathcal{H}_\infty$ norm, amongst others, are very popular and useful notions to formulate performance specifications [7, 8]. Methods for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control have emerged since the 1980s [9], and still constitute an active research branch. Both the single- and multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ analysis problems for linear time-invariant (LTI) systems can directly be formulated as LMI optimization problems. The corresponding synthesis problems, on the other hand, generally involve bilinear matrix inequalities (BMIs) which, as opposed to LMIs, are nonconvex and therefore hard to solve [10–13]. For the synthesis of single-objective $\mathcal{H}_2$ or $\mathcal{H}_\infty$ full-order dynamic output feedback controllers for LTI systems, the BMI condition can be reformulated into an equivalent LMI by a nonlinear change of variables [2, 14] or elimination of the controller variables [4, 15].

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As it is often complicated to capture all design specifications in a single objective, several approaches to multi-objective controller design have been developed [2, 14, 16–18], allowing the incorporation of various $\mathcal{H}_2$ and $\mathcal{H}_\infty$ (as well as other) performance specifications. Specifically, the design of full-order multi-objective LTI controllers gives rise to a nonconvex problem, and conservatism is introduced in the aforementioned approaches to derive convex sufficient conditions. Applying the nonlinear transformation of controller variables that is presented in [2, 14], sufficient LMIs result when selecting a single Lyapunov matrix for all performance specifications. This conservative procedure is named the Lyapunov shaping paradigm. A less conservative approach is proposed in [18] for the discrete-time case, relying on the introduction of a slack variable G. Products between the closed-loop matrices and G appear in the resulting LMIs, while the closed-loop matrices and the Lyapunov matrices are decoupled. Selecting a constant matrix G in the latter synthesis method, which is referred to as the G shaping paradigm, different Lyapunov matrices are allowed for each performance channel while convexity is retained.

The existence of a convex reformulation for the design of reduced-order $\mathcal{H}_2/\mathcal{H}_\infty$ controllers is unknown, even when only one performance specification is imposed [2, 13, 19]. Despite the lack of such a convex condition, several approaches have been developed for reduced-order controller design [20]. Those include solving the BMI problem directly [19], solving a nonconvex reformulation in terms of an LMI plus a rank constraint [21–23], or setting up convex sufficient conditions [24–29].

In this paper, an approach is presented that falls into the latter category. The approach is based on the idea proposed in [24, 25, 30, 31] and the works [27, 32, 33]. In the latter, an LMI procedure consisting of two stages is presented to design reduced-order controllers such that the closed-loop system satisfies one or more $\mathcal{H}_2/\mathcal{H}_\infty$ performance specifications. In this procedure, a stabilizing state feedback for an augmented system is designed first and subsequently used as a parameter in a sufficient LMI condition for reduced-order controller design. A reduced-order controller results whenever the LMI optimization problem is feasible. However, the state feedback design in the first step has infinitely many solutions, and the particular solution selected strongly affects the subsequent reduced-order controller design step. In fact, structural constraints need to be imposed on the LMI variables in the state-feedback design step to avoid synthesis of a reduced-order controller that is reducible to a static output feedback [27]. As selecting an appropriate state feedback (in the first step) to obtain satisfactory results in the second step is not trivial, we propose a novel approach starting from a set of full-order $\mathcal{H}_2/\mathcal{H}_\infty$ controllers (for the original system), and explain how to intuitively select this set of full-order controllers. Given such a set of feedback controllers, it is demonstrated how to compute a reduced-order $\mathcal{H}_2/\mathcal{H}_\infty$ controller by solving a set of sufficient LMIs. The computation of a reduced-order controller can either be performed in one step or by successive reduction of the controller order. The design of single-objective $\mathcal{H}_2$ or $\mathcal{H}_\infty$ controllers follows as a trivial special case. Some less trivial extensions are discussed in the main part of the paper.

The main contributions of this paper are summarized as follows. Firstly, a novel projection lemma based convex framework for reduced-order multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ controller design is presented, treating continuous time and discrete time in a unified way. This contrasts our earlier contributions [34, 35], that only handle discrete-time systems. In addition, compared to [34, 35], extra degrees of freedom are added in the LMI conditions to allow a reduction of conservatism. Secondly, since the Lyapunov and G shaping paradigms for full-order multi-objective controller design give rise to sufficient conditions, a reduction of conservatism for these designs using the proposed reduced-order synthesis LMIs is motivated. It is theoretically and numerically shown that reductions of conservatism can be achieved, by using a different full-order controller parameter for each performance specification.

The paper is organized as follows. First, the problem is mathematically formulated and the required notation is introduced in Section 2. Subsequently, a detailed description of the reduced-order multi-objective controller design approach is given in Section 3. Then, Section 4 discusses three numerical examples, confirming the potential of the proposed LMI approach compared to existing methods for reduced-order controller design. Finally, the conclusions are given in Section 5.

Notation. Since different controller orders are used throughout this paper, the dependence of a matrix X on the controller order q is indicated by $X^{(q)}$ whenever necessary. I_n denotes the identity matrix of dimension n and $0_{m\times n}$ denotes a zero matrix of dimension $m \times n$. The subscripts are omitted whenever the dimensions can be inferred from the context. The (Hermitian) transpose of a matrix X (or $X^{(q)}$) is written as X' (or $X^{(q)\prime}$). The dependence of matrix X (or $X^{(q)}$) on a performance channel j is denoted by X_j (or $X_j^{(q)}$). The set of $n \times n$ real symmetric matrices is defined as $\mathbb{S}^n$. A star ($\star$) indicates symmetric terms in matrix inequalities. The symbol $\otimes$ denotes the matrix Kronecker product. The notation $\mathrm{He}\{X\} := X + X'$ is used. Finally, $X_\perp$ is an arbitrary matrix whose colums form a basis for the nullspace of X.

2. Problem formulation

Consider the finite-dimensional LTI state-space realization

$$\left\{ \begin{array}{rcl} \delta x & = & Ax + B_w w + B_u u, \\ z & = & C_z x + D_{zw} w + D_{zu} u, \\ y & = & C_y x + D_{yw} w, \end{array} \right. \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ the control input, $w \in \mathbb{R}^{n_w}$ the exogenous input, $z \in \mathbb{R}^{n_z}$ the regulated output, and $y \in \mathbb{R}^{n_y}$ the measured output. In discrete time, δ denotes the forward shift operator, while $\delta x = \frac{dx}{dt}$ in continuous time. The direct feedthrough matrix D_{yu} is set to 0 without loss of generality [15]. The objective in this paper is to compute a dynamic output feedback controller of fixed order q, where $0 \leq q < n_x$, that stabilizes (1) and meets various closed-loop performance specifications. Such a controller is written as

$$\left\{ \begin{array}{rcl} \delta x_c & = & A_c x_c + B_c y, \\ u & = & C_c x_c + D_c y, \end{array} \right. \tag{2}$$

where $x_c \in \mathbb{R}^q$. Whenever $q = n_x$, the controller is of full order, whereas $q < n_x$ corresponds to a reduced-order controller. Setting $q = 0$ results in a static output feedback. Interconnecting system (1) with a controller (2) yields the closed-loop dynamics

$$H : \left\{ \begin{array}{rcl} \delta \tilde{x} & = & \mathcal{A}\tilde{x} + \mathcal{B}w, \\ z & = & \mathcal{C}\tilde{x} + \mathcal{D}w, \end{array} \right. \tag{3}$$

where $\tilde{x} = \left[\begin{array}{cc} x' & x_c' \end{array} \right]' \in \mathbb{R}^{n_x+q}$ is a closed-loop state vector. Defining the matrices

$$\left[\begin{array}{ccc} \tilde{A} & \tilde{B}_w & \tilde{B}_u \\ \tilde{C}_z & \tilde{D}_{zw} & \tilde{D}_{zu} \\ \tilde{C}_y & \tilde{D}_{yw} & 0 \end{array} \right] := \left[\begin{array}{cc|c|cc} A & 0 & B_w & 0 & B_u \\ 0 & 0 & 0 & I_q & 0 \\ \hline C_z & 0 & D_{zw} & 0 & D_{zu} \\ \hline 0 & I_q & 0 & 0 & 0 \\ C_y & 0 & D_{yw} & 0 & 0 \end{array} \right] \tag{4}$$

and the controller parameter

$$\Theta := \left[\begin{array}{cc} A_c & B_c \\ C_c & D_c \end{array} \right], \tag{5}$$

the affine dependence of the closed-loop matrices of H on Θ is expressed as

$$\left[\begin{array}{cc} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{array} \right] = \left[\begin{array}{cc} \tilde{A} & \tilde{B}_w \\ \tilde{C}_z & \tilde{D}_{zw} \end{array} \right] + \left[\begin{array}{c} \tilde{B}_u \\ \tilde{D}_{zu} \end{array} \right] \Theta \left[\begin{array}{cc} \tilde{C}_y & \tilde{D}_{yw} \end{array} \right]. \tag{6}$$

The dependence of the matrices in (6) on the controller order q can be inferred from (4) and (5), and is specified by a superscript (q) whenever needed.

The various performance specifications are labeled by the index j, where j is a positive integer. Hence, the closed-loop dynamics corresponding to performance channel j are denoted by

$$H_j : \left\{ \begin{array}{rcl} \delta \tilde{x} & = & \mathcal{A}\tilde{x} + \mathcal{B}_j w_j, \\ z_j & = & \mathcal{C}_j\tilde{x} + \mathcal{D}_j w_j, \end{array} \right. \tag{7}$$

where appropriate selection matrices L_j and R_j are defined such that $\mathcal{B}_j = \mathcal{B}R_j$, $\mathcal{C}_j = L_j\mathcal{C}$ and $\mathcal{D}_j = L_j\mathcal{D}R_j$. Note that substitution of $w = R_j w_j$ and $z_j = L_j z$ in (3) results in (7). The dimension of z_j is denoted by n_{zj}. The affine dependence of the system matrices of H_j on Θ can be expressed in accordance with (6), by defining the matrices $\tilde{B}_j := \tilde{B}_w R_j$, $\tilde{C}_j := L_j\tilde{C}_z$, $\tilde{D}_j := L_j\tilde{D}_{zw}R_j$, $\tilde{D}_{ju} := L_j\tilde{D}_{zu}$ and $\tilde{D}_{yj} := \tilde{D}_{yw}R_j$. Whenever necessary, the dependence of H_j on a controller parameterized by Θ is indicated as

$$H_j(\Theta) : \left\{ \begin{array}{rcl} \delta\tilde{x} & = & \mathcal{A}(\Theta)\tilde{x} + \mathcal{B}_j(\Theta)w_j, \\ z_j & = & \mathcal{C}_j(\Theta)\tilde{x} + \mathcal{D}_j(\Theta)w_j. \end{array} \right. \tag{8}$$

The set of performance indices j is denoted by $\mathcal{S}$, and the sets of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indices are denoted by $\mathcal{S}_{\mathcal{H}_2}$ respectively $\mathcal{S}_{\mathcal{H}_\infty}$. Since only $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance channels are considered in this paper, $\mathcal{S} = \mathcal{S}_{\mathcal{H}_2} \cup \mathcal{S}_{\mathcal{H}_\infty}$. For notational convenience, it is assumed that $\mathcal{S}_{\mathcal{H}_2} \cap \mathcal{S}_{\mathcal{H}_\infty} = \emptyset$ without loss of generality, and a general objective function

$$\sum_{j\in\mathcal{S}_{\mathcal{H}_2}} a_j\mu_j + \sum_{j\in\mathcal{S}_{\mathcal{H}_\infty}} a_j\gamma_j, \quad \text{with } a_j \geq 0 \text{ for } j \in \mathcal{S}, \tag{9}$$

is considered, where μ_j and γ_j denote upper bounds on $\|H_j\|_2^2$ respectively $\|H_j\|_\infty^2$, $j \in \mathcal{S}$, and $a_j = 0$ if j is not an optimization channel. Upper bounds $b_j > 0$, $j \in \mathcal{S}$ can be taken into account by imposing the constraints

$$\mu_j < b_j, \quad j \in \mathcal{S}_{\mathcal{H}_2} \quad \text{and} \quad \gamma_j < b_j, \quad j \in \mathcal{S}_{\mathcal{H}_\infty},$$

where $b_j = \infty$ if channel j is unconstrained.

3. Reduced-order multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ controller design

This section presents a novel projection lemma based convex framework for analysis and synthesis of reduced-order multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ controllers for LTI systems. Continuous time and discrete time are treated in a unified fashion. First, new extended LMI characterizations for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance are presented in Subsection 3.1, which are adapted to reduced-order multi-objective synthesis LMIs in Subsection 3.2. Subsequently, the conservatism of the novel LMIs is discussed in detail, followed by two extensions of the reduced-order synthesis conditions.

3.1. Extended LMIs for $\mathcal{H}_2/\mathcal{H}_\infty$ performance analysis

The unified formulation of the continuous-time and discrete-time performance characterizations, see for example [1, 3], is based on the general description of the stability region

$$\Omega(\Phi) = \left\{ \lambda \in \mathbb{C} \;\middle|\; \begin{bmatrix} 1 \\ \lambda \end{bmatrix}' \Phi \begin{bmatrix} 1 \\ \lambda \end{bmatrix} < 0 \right\} \tag{10}$$

with $\Phi \in \mathbb{S}^2$. The open left half plane and the open unit disk are characterized by selecting $\Phi = \Phi_\mathrm{c}$, respectively, $\Phi = \Phi_\mathrm{d}$, where

$$\Phi_\mathrm{c} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Phi_\mathrm{d} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{11}$$

While the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance of a given closed-loop system H_j, $j \in \mathcal{S}$, can directly be analyzed using well-known LMI conditions, see for example [1, 3, 4, 36], this subsection provides extended LMI characterizations that form the starting point for the derivation of the reduced-order synthesis LMIs presented in Subsection 3.2. Firstly, the extended LMIs rely on the use of an augmented controller parameter $\Theta_\mathrm{a} \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_y)}$, that is constructed from $\Theta \in \mathbb{R}^{(q+n_u)\times(q+n_y)}$ by adding Hurwitz ($\Phi = \Phi_\mathrm{c}$) or Schur ($\Phi = \Phi_\mathrm{d}$) stable uncontrollable and/or unobservable dynamics. Hence, Θ_a and Θ have identical closed-loop stability and performance properties. Secondly, defining arbitrary full-order controller parameters Λ_j, $j \in \mathcal{S}$, the closed-loop analysis of $H_j(\Theta_\mathrm{a})$ is linked to the closed-loop analysis of $H_j(\Lambda_j)$. Namely, noting that Λ_j and Θ_a have the same dimensions, and using (6), the following relation between the closed-loop matrices of $H_j(\Theta_\mathrm{a})$ and the closed-loop matrices of $H_j(\Lambda_j)$ is obtained:

$$\begin{bmatrix} \mathcal{A}(\Theta_\mathrm{a}) & \mathcal{B}_j(\Theta_\mathrm{a}) \\ \mathcal{C}_j(\Theta_\mathrm{a}) & \mathcal{D}_j(\Theta_\mathrm{a}) \end{bmatrix} = \begin{bmatrix} \mathcal{A}(\Lambda_j) & \mathcal{B}_j(\Lambda_j) \\ \mathcal{C}_j(\Lambda_j) & \mathcal{D}_j(\Lambda_j) \end{bmatrix} + \begin{bmatrix} \tilde{B}_u^{(n_x)} \\ \tilde{D}_{ju}^{(n_x)} \end{bmatrix} (\Theta_\mathrm{a} - \Lambda_j) \begin{bmatrix} \tilde{C}_y^{(n_x)} & \tilde{D}_{yj}^{(n_x)} \end{bmatrix}, \quad j \in \mathcal{S}. \tag{12}$$

It is remarked that (12) is valid for any full-order controller parameter Λ_j, hence also unstable and destabilizing Λ_j are allowed at this point. Relation (12) gives rise to the extended $\mathcal{H}_2$ performance characterization presented in Theorem 1.

Theorem 1 (Extended $\mathcal{H}_2$ performance)**.** *Let $\Lambda_j \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_y)}$ be an arbitrary matrix, and let $\Theta_\mathrm{a} \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_y)}$ denote a given controller parameter that is constructed from $\Theta \in \mathbb{R}^{(q+n_u)\times(q+n_y)}$ by adding Hurwitz ($\Phi = \Phi_\mathrm{c}$) or Schur ($\Phi = \Phi_\mathrm{d}$) stable uncontrollable and/or unobservable dynamics. Furthermore, assume that $\mathcal{D}_j(\Theta) = 0$. Then, the closed-loop system $H_j(\Theta)$ defined in* (8) *is exponentially stable and $\|H_j(\Theta)\|_2^2 < \mu_j$ if, and only if, there exist matrices $P_j \in \mathbb{S}^{2n_x}$, $W_j \in \mathbb{S}^{n_{zj}}$, $X_{1j} \in \mathbb{R}^{2n_x\times(n_x+n_u)}$, $X_{2j} \in \mathbb{R}^{n_w\times(n_x+n_u)}$, $X_{3j} \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_u)}$, $X_{4j} \in \mathbb{R}^{n_{zj}\times(n_x+n_u)}$, $X_{5j} \in \mathbb{R}^{2n_x\times(n_x+n_u)}$ and $X_{6j} \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_u)}$ such that* $\mathrm{trace}(W_j) < \mu_j$ *and the following LMIs hold*

$$\left[\begin{array}{cc|c} I & 0 & 0 \\ \mathcal{A}(\Lambda_j) & \mathcal{B}_j(\Lambda_j) & \tilde{B}_u^{(n_x)} \\ 0 & I & 0 \end{array}\right]' \begin{bmatrix} \Phi \otimes P_j & 0 \\ 0 & -I \end{bmatrix} \left[\begin{array}{cc|c} I & 0 & 0 \\ \mathcal{A}(\Lambda_j) & \mathcal{B}_j(\Lambda_j) & \tilde{B}_u^{(n_x)} \\ 0 & I & 0 \end{array}\right] \\ + \mathrm{He}\left\{ \begin{bmatrix} X_{1j} \\ X_{2j} \\ X_{3j} \end{bmatrix} \begin{bmatrix} (\Theta_\mathrm{a} - \Lambda_j)\tilde{C}_y^{(n_x)} & (\Theta_\mathrm{a} - \Lambda_j)\tilde{D}_{yj}^{(n_x)} & -I \end{bmatrix} \right\} \prec 0, \tag{13a}$$

$$\left[\begin{array}{cc|c} W_j & \mathcal{C}_j(\Lambda_j) & \tilde{D}_{ju}^{(n_x)} \\ \star & P_j & 0 \\ \hline \star & \star & 0 \end{array}\right] + \mathrm{He}\left\{ \begin{bmatrix} X_{4j} \\ X_{5j} \\ X_{6j} \end{bmatrix} \begin{bmatrix} 0 & (\Theta_\mathrm{a} - \Lambda_j)\tilde{C}_y^{(n_x)} & -I \end{bmatrix} \right\} \succ 0, \tag{13b}$$

where $\Phi = \Phi_\mathrm{c}$ or $\Phi = \Phi_\mathrm{d}$ as in (11) *for the continuous, respectively, discrete-time conditions.*

Proof. *Deriving the matrices*

$$\begin{bmatrix} (\Theta_\mathrm{a} - \Lambda_j)\tilde{C}_y^{(n_x)} & (\Theta_\mathrm{a} - \Lambda_j)\tilde{D}_{yj}^{(n_x)} & -I \end{bmatrix}_\perp = \begin{bmatrix} I & 0 \\ 0 & I \\ (\Theta_\mathrm{a} - \Lambda_j)\tilde{C}_y^{(n_x)} & (\Theta_\mathrm{a} - \Lambda_j)\tilde{D}_{yj}^{(n_x)} \end{bmatrix},$$

$$\begin{bmatrix} 0 & (\Theta_\mathrm{a} - \Lambda_j)\tilde{C}_y^{(n_x)} & -I \end{bmatrix}_\perp = \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & (\Theta_\mathrm{a} - \Lambda_j)\tilde{C}_y^{(n_x)} \end{bmatrix},$$

applying the projection lemma [4] on (13a) *and* (13b)*, and using* (12) *gives the equivalent conditions*

$$\begin{bmatrix} I & 0 \\ \mathcal{A}(\Theta_\mathrm{a}) & \mathcal{B}_j(\Theta_\mathrm{a}) \\ 0 & I \end{bmatrix}' \begin{bmatrix} \Phi \otimes P_j & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathcal{A}(\Theta_\mathrm{a}) & \mathcal{B}_j(\Theta_\mathrm{a}) \\ 0 & I \end{bmatrix} \prec 0, \quad \begin{bmatrix} W_j & \mathcal{C}_j(\Theta_\mathrm{a}) \\ \star & P_j \end{bmatrix} \succ 0, \tag{14}$$

which is the standard LMI characterization for the $\mathcal{H}_2$ performance of $H_j(\Theta_\mathrm{a})$ [1]. The fact that Θ_a has the same stability and performance properties as Θ finishes the proof.

Remark 1. *In case of a discrete-time system that is not strictly proper (i.e. $\mathcal{D}_j(\Theta) \neq 0$),* (13b) *should be* *replaced by the LMI*

$$\left[\begin{array}{ccc|c} W_j & \mathcal{C}_j(\Lambda_j) & \mathcal{D}_j(\Lambda_j) & \tilde{D}_{ju}^{(n_x)} \\ \star & P_j & 0 & 0 \\ \star & \star & I & 0 \\ \hline \star & \star & \star & 0 \end{array}\right] + \mathrm{He}\left\{\begin{bmatrix} X_{4j} \\ X_{5j} \\ X_{6j} \\ X_{7j} \end{bmatrix}\begin{bmatrix} 0 & (\Theta_\mathrm{a}-\Lambda_j)\tilde{C}_y^{(n_x)} & (\Theta_\mathrm{a}-\Lambda_j)\tilde{D}_{yj}^{(n_x)} & -I \end{bmatrix}\right\} \succ 0.$$

A novel extended $\mathcal{H}_\infty$ performance characterization is provided in Theorem 2.

Theorem 2 (Extended $\mathcal{H}_\infty$ performance)**.** *Let $\Lambda_j \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_y)}$ be an arbitrary matrix, and let $\Theta_\mathrm{a} \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_y)}$ denote a given controller parameter that is constructed from $\Theta \in \mathbb{R}^{(q+n_u)\times(q+n_y)}$ by adding Hurwitz ($\Phi = \Phi_\mathrm{c}$) or Schur ($\Phi = \Phi_\mathrm{d}$) stable uncontrollable and/or unobservable dynamics. Then, the closed-loop system $H_j(\Theta)$ defined in* (8) *is exponentially stable and $\|H_j(\Theta)\|_\infty^2 < \gamma_j$ if, and only if, there exist matrices $P_j \in \mathbb{S}^{2n_x}$, $X_{1j} \in \mathbb{R}^{2n_x\times(n_x+n_u)}$, $X_{2j} \in \mathbb{R}^{n_w\times(n_x+n_u)}$ and $X_{3j} \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_u)}$ such that the following LMIs hold*

$$P_j \succ 0, \tag{15a}$$

$$\left[\begin{array}{cc|c} I & 0 & 0 \\ \mathcal{A}(\Lambda_j) & \mathcal{B}_j(\Lambda_j) & \tilde{B}_u^{(n_x)} \\ 0 & I & 0 \\ \mathcal{C}_j(\Lambda_j) & \mathcal{D}_j(\Lambda_j) & \tilde{D}_{ju}^{(n_x)} \end{array}\right]' \begin{bmatrix} \Phi\otimes P_j & 0 & 0 \\ 0 & -\gamma_j I & 0 \\ 0 & 0 & I \end{bmatrix} \left[\begin{array}{cc|c} I & 0 & 0 \\ \mathcal{A}(\Lambda_j) & \mathcal{B}_j(\Lambda_j) & \tilde{B}_u^{(n_x)} \\ 0 & I & 0 \\ \mathcal{C}_j(\Lambda_j) & \mathcal{D}_j(\Lambda_j) & \tilde{D}_{ju}^{(n_x)} \end{array}\right] + \mathrm{He}\left\{\begin{bmatrix} X_{1j} \\ X_{2j} \\ X_{3j} \end{bmatrix}\begin{bmatrix} (\Theta_\mathrm{a}-\Lambda_j)\tilde{C}_y^{(n_x)} & (\Theta_\mathrm{a}-\Lambda_j)\tilde{D}_{yj}^{(n_x)} & -I \end{bmatrix}\right\} \prec 0, \tag{15b}$$

where $\Phi = \Phi_\mathrm{c}$ or $\Phi = \Phi_\mathrm{d}$ as in (11) *for the continuous, respectively, discrete-time conditions.*

The proof of Theorem 2 is a straightforward extension of the proof of Theorem 1, and is therefore omitted. It should be mentioned that, for any value of Λ_j, Theorem 1 and Theorem 2 provide necessary and sufficient conditions for the $\mathcal{H}_2$, respectively, $\mathcal{H}_\infty$ performance of the closed-loop system $H_j(\Theta)$ as long as no structure is enforced on $X_{1j}, \dots, X_{6j}$.

In order to arrive at reduced-order synthesis LMIs in the next subsection, we need to impose the structural constraints $X_{1j} = 0$, $X_{2j} = 0$, $X_{4j} = 0$ and $X_{5j} = 0$ in the conditions (13a), (13b) and (15b). It is worth remarking that the resulting conditions feature a clear interpretation. Namely, applying the projection lemma on (13a) and (13b) gives the equivalent conditions (14) together with the additional constraints

$$\begin{bmatrix} I & 0 \\ \mathcal{A}(\Lambda_j) & \mathcal{B}_j(\Lambda_j) \\ 0 & I \end{bmatrix}' \begin{bmatrix} \Phi\otimes P_j & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathcal{A}(\Lambda_j) & \mathcal{B}_j(\Lambda_j) \\ 0 & I \end{bmatrix} \prec 0, \quad \begin{bmatrix} W_j & \mathcal{C}_j(\Lambda_j) \\ \star & P_j \end{bmatrix} \succ 0,$$

implying that Λ_j should be stabilizing, and such that $\|H_j(\Lambda_j)\|_2^2 < \mu_j$, $j \in \mathcal{S}_{\mathcal{H}_2}$. In a similar fashion, selecting $X_{1j} = 0$ and $X_{2j} = 0$ in (15b) implies that Λ_j should be a stabilizing controller satisfying $\|H_j(\Lambda_j)\|_\infty^2 < \gamma_j$, $j \in \mathcal{S}_{\mathcal{H}_\infty}$. Note that the choice for a particular controller augmentation, i.e. the selection of Θ_a, influences the LMI conditions. Moreover, P_j should be a Lyapunov certificate both for $H_j(\Theta_\mathrm{a})$ and $H_j(\Lambda_j)$, $j \in \mathcal{S}$. Imposing structural constraints is necessary in many LMI-based controller synthesis approaches, see for example [18, 37]. The fact that the reduced-order controller must share the same Lyapunov matrix with the auxiliary controllers makes our approach comparable to the approaches [24, 25, 27]. The latter approaches rely on a common Lyapunov certificate for the closed-loop systems corresponding to a state feedback and a static output-feedback controller.

3.2. Sufficient LMIs for reduced-order multi-objective controller design

This subsection presents a general framework of sufficient LMIs for reduced-order multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ controller design, which is based on the novel extended $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance conditions from Subsection 3.1. The LMIs require a set of stabilizing full-order controllers, which are parameterized by

$$\Lambda_j \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_y)}, \quad j \in \mathcal{S}$$

as in (5). These controllers should be computed a priori using, for example, the approaches discussed in [2, 18].

Sufficient LMIs for the design of a stabilizing reduced-order controller with a guaranteed upper bound on the $\mathcal{H}_2$ performance of the closed-loop system (7), $j \in \mathcal{S}_{\mathcal{H}_2}$, are presented in Theorem 3.

Theorem 3 (Reduced-order $\mathcal{H}_2$ controller design)**.** *Assume that $\tilde{D}_j = 0$, and that $\tilde{D}_{ju} = 0$ or $\tilde{D}_{yj} = 0$. Let Λ_j, $j \in \mathcal{S}_{\mathcal{H}_2}$, parameterize a stabilizing full-order controller for system (1), as in (5), and let $\mathcal{A}(\Lambda_j)$, $\mathcal{B}_j(\Lambda_j)$, $\mathcal{C}_j(\Lambda_j)$ and $\mathcal{D}_j(\Lambda_j)$ denote the corresponding closed-loop matrices for performance channel j, as in (8). For a given integer q ($0 \leq q < n_x$), let the matrices $U \in \mathbb{R}^{(q+n_u)\times(n_x+n_u)}$ and $V \in \mathbb{R}^{(q+n_y)\times(n_x+n_y)}$ be defined as*

$$U = \begin{bmatrix} I_q & 0_{q\times(n_x-q)} & 0 \\ 0 & 0 & I_{n_u} \end{bmatrix}, \quad V = \begin{bmatrix} I_q & 0_{q\times(n_x-q)} & 0 \\ 0 & 0 & I_{n_y} \end{bmatrix}.$$

Let $A_{22} \in \mathbb{R}^{(n_x-q)\times(n_x-q)}$ be a given matrix with all of its eigenvalues in the stability region $\Omega(\Phi)$, as in (10). If there exist matrices $P_j \in \mathbb{S}^{2n_x}$, $W_j \in \mathbb{S}^{n_{zj}}$, $\hat{\Theta} \in \mathbb{R}^{(q+n_u)\times(n_x+n_y)}$ and

$$Y_j = \begin{bmatrix} Y_{11} & Y_{12j} & Y_{13} \\ 0 & Y_{22j} & 0 \\ Y_{31} & Y_{32j} & Y_{33} \end{bmatrix}$$

with $Y_{11} \in \mathbb{R}^{q\times q}$, $Y_{22j} \in \mathbb{R}^{(n_x-q)\times(n_x-q)}$, $Y_{33} \in \mathbb{R}^{n_u\times n_u}$, and a scalar μ_j such that $\mathrm{trace}(W_j) < \mu_j$ and the following LMIs hold

$$\left[\begin{array}{cc|c} I & 0 & 0 \\ \mathcal{A}(\Lambda_j) & \mathcal{B}_j(\Lambda_j) & \tilde{B}_u^{(n_x)} \\ 0 & I & 0 \end{array}\right]' \begin{bmatrix} \Phi\otimes P_j & 0 \\ 0 & -I \end{bmatrix} \left[\begin{array}{cc|c} I & 0 & 0 \\ \mathcal{A}(\Lambda_j) & \mathcal{B}_j(\Lambda_j) & \tilde{B}_u^{(n_x)} \\ 0 & I & 0 \end{array}\right] + \mathrm{He}\left\{ \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \begin{bmatrix} Z_j\tilde{C}_y^{(n_x)} & Z_j\tilde{D}_{yj}^{(n_x)} & -Y_j \end{bmatrix} \right\} \prec 0, \quad (16a)$$

$$\left[\begin{array}{cc|c} W_j & \mathcal{C}_j(\Lambda_j) & \tilde{D}_{ju}^{(n_x)} \\ \star & P_j & 0 \\ \hline \star & \star & 0 \end{array}\right] + \mathrm{He}\left\{ \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \begin{bmatrix} 0 & -Z_j\tilde{C}_y^{(n_x)} & Y_j \end{bmatrix} \right\} \succ 0, \quad (16b)$$

where Z_j is given by

$$Z_j := U'\hat{\Theta} + Y_j\left(\begin{bmatrix} 0_{q\times q} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & 0_{n_u\times n_y} \end{bmatrix} - \Lambda_j \right),$$

then the reduced-order controller parameterized by

$$\Theta := \begin{bmatrix} Y_{11} & Y_{13} \\ Y_{31} & Y_{33} \end{bmatrix}^{-1} \hat{\Theta}V' \quad (17)$$

stabilizes the closed-loop system (7) with a guaranteed upper bound on the closed-loop $\mathcal{H}_2$ performance $\|H_j\|_2^2 < \mu_j$.

Proof. *Assume that the LMIs* (16a) *and* (16b) *are feasible. First, define the following augmented controller parameter*

$$\Theta_{\mathrm{a}} := \left[\begin{array}{cc|c} A_c & A_{12} & B_c \\ 0 & A_{22} & 0 \\ \hline C_c & C_2 & D_c \end{array}\right],$$

corresponding to a Kalman canonical form with uncontrollable part. Substituting $\Theta = \Theta_{\mathrm{a}}$ and $q = n_x$ in (6)*, selecting performance channel $j \in \mathcal{S}_{\mathcal{H}_2}$ yields the closed-loop matrices*

$$\begin{bmatrix} \mathcal{A}(\Theta_{\mathrm{a}}) & \mathcal{B}_j(\Theta_{\mathrm{a}}) \\ \mathcal{C}_j(\Theta_{\mathrm{a}}) & \mathcal{D}_j(\Theta_{\mathrm{a}}) \end{bmatrix} = \begin{bmatrix} \tilde{A}^{(n_x)} & \tilde{B}_j^{(n_x)} \\ \tilde{C}_j^{(n_x)} & \tilde{D}_j \end{bmatrix} + \begin{bmatrix} \tilde{B}_u^{(n_x)} \\ \tilde{D}_{ju}^{(n_x)} \end{bmatrix} \Theta_{\mathrm{a}} \begin{bmatrix} \tilde{C}_y^{(n_x)} & \tilde{D}_{yj}^{(n_x)} \end{bmatrix}. \tag{18}$$

Note that $\mathcal{D}_j(\Theta_{\mathrm{a}}) = 0$, since $\tilde{D}_j = 0$ and $\tilde{D}_{ju}^{(n_x)} = 0$ or $\tilde{D}_{yj}^{(n_x)} = 0$. As A_{22} is stable with respect to the stability region $\Omega(\Phi)$, the controllers parameterized by Θ and Θ_{a} have the same stability properties and input-output behaviour, and thus yield the same closed-loop performance. Therefore, according to Theorem 1, it suffices to show that the matrix inequalities (13) *hold. To show that* (13) *indeed holds, note that*

$$Z_j = Y_j(\Theta_{\mathrm{a}} - \Lambda_j), \tag{19}$$

with

$$\hat{\Theta} := \begin{bmatrix} Y_{11} & Y_{13} \\ Y_{31} & Y_{33} \end{bmatrix} \begin{bmatrix} A_c & A_{12} & B_c \\ C_c & C_2 & D_c \end{bmatrix}, \tag{20}$$

and that (20) *results in*

$$\Theta = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}$$

after application of the nonlinear transformation (17)*. Substituting* (19) *in the LMIs* (16a) *and* (16b) *results in* (13) *with the specific selection*

$$\begin{bmatrix} X_{1j} \\ X_{2j} \\ X_{3j} \end{bmatrix} := \begin{bmatrix} 0 \\ 0 \\ Y_j \end{bmatrix}, \quad \begin{bmatrix} X_{4j} \\ X_{5j} \\ X_{6j} \end{bmatrix} := \begin{bmatrix} 0 \\ 0 \\ -Y_j \end{bmatrix}. \tag{21}$$

Note that (21) *implies that $H_j(\Lambda_j)$ should be exponentially stable and satisfy the $\mathcal{H}_2$ performance bound $\|H_j(\Lambda_j)\|_2^2 < \mu_j$, see also Remark 3. This finishes the proof.*

Remark 2. *The assumptions $\tilde{D}_j = 0$, and $\tilde{D}_{ju} = 0$ or $\tilde{D}_{yj} = 0$, assure a finite $\mathcal{H}_2$ norm of the closed-loop system in continuous time ($\Phi = \Phi_{\mathrm{c}}$). While the reduced-order synthesis LMIs require that $\tilde{D}_j = 0$, the case $\tilde{D}_{ju} \neq 0$ and $\tilde{D}_{yj} \neq 0$ can be handled as follows:*

1. *Design a* strictly proper *stabilizing full-order controller, parameterized by Λ_j.*
2. *Subdivide the LMI variable $\hat{\Theta}$ as follows*

$$\hat{\Theta} := \begin{bmatrix} \hat{\Theta}_{11} & \hat{\Theta}_{12} \\ \hat{\Theta}_{21} & \hat{\Theta}_{22} \end{bmatrix}, \tag{22}$$

with $\hat{\Theta}_{22} \in \mathbb{R}^{n_u \times n_y}$, and impose the constraints $\hat{\Theta}_{22} = 0$ and $Y_{31} = 0$.

In the discrete-time case ($\Phi = \Phi_{\mathrm{d}}$), $\tilde{D}_j$, $\tilde{D}_{ju}$ and $\tilde{D}_{yj}$ are all allowed to be nonzero. In that case (16b) *should* *be replaced by*

$$\left[\begin{array}{ccc|c} W_j & \mathcal{C}_j(\Lambda_j) & \mathcal{D}_j(\Lambda_j) & \tilde{D}_{ju}^{(n_x)} \\ \star & P_j & 0 & 0 \\ \star & \star & I & 0 \\ \hline \star & \star & \star & 0 \end{array}\right] + \mathrm{He}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix} \begin{bmatrix} 0 & -Z_j\tilde{C}_y^{(n_x)} & -Z_j\tilde{D}_{yj}^{(n_x)} & Y_j \end{bmatrix} \right\} \succ 0.$$

The following theorem presents sufficient LMI conditions for the design of a stabilizing reduced-order controller with a guaranteed upper bound on the $\mathcal{H}_\infty$ performance of the closed-loop system (7), $j \in \mathcal{S}_{\mathcal{H}_\infty}$.

Theorem 4 (Reduced-order $\mathcal{H}_\infty$ controller design)**.** *Let Λ_j, $j \in \mathcal{S}_{\mathcal{H}_\infty}$, parameterize a stabilizing full-order controller for system* (1)*, as in* (5)*, and let $\mathcal{A}(\Lambda_j)$, $\mathcal{B}_j(\Lambda_j)$, $\mathcal{C}_j(\Lambda_j)$ and $\mathcal{D}_j(\Lambda_j)$ denote the corresponding closed-loop matrices for performance channel j, as in* (8)*. For a given integer q ($0 \leq q < n_x$), let the matrices $U \in \mathbb{R}^{(q+n_u)\times(n_x+n_u)}$ and $V \in \mathbb{R}^{(q+n_y)\times(n_x+n_y)}$ be defined as*

$$U = \begin{bmatrix} I_q & 0_{q\times(n_x-q)} & 0 \\ 0 & 0 & I_{n_u} \end{bmatrix}, \quad V = \begin{bmatrix} I_q & 0_{q\times(n_x-q)} & 0 \\ 0 & 0 & I_{n_y} \end{bmatrix}.$$

Let $A_{22} \in \mathbb{R}^{(n_x-q)\times(n_x-q)}$ be a given matrix with all of its eigenvalues in the stability region $\Omega(\Phi)$, as in (10)*. If there exist matrices $P_j \in \mathbb{S}^{2n_x}$, $\hat{\Theta} \in \mathbb{R}^{(q+n_u)\times(n_x+n_y)}$ and*

$$Y_j = \begin{bmatrix} Y_{11} & Y_{12j} & Y_{13} \\ 0 & Y_{22j} & 0 \\ Y_{31} & Y_{32j} & Y_{33} \end{bmatrix}$$

with $Y_{11} \in \mathbb{R}^{q\times q}$, $Y_{22j} \in \mathbb{R}^{(n_x-q)\times(n_x-q)}$, $Y_{33} \in \mathbb{R}^{n_u\times n_u}$, and a scalar γ_j such that the following LMIs hold

$$P_j \succ 0, \tag{23a}$$

$$\left[\begin{array}{cc|c} I & 0 & 0 \\ \mathcal{A}(\Lambda_j) & \mathcal{B}_j(\Lambda_j) & \tilde{B}_u^{(n_x)} \\ 0 & I & 0 \\ \mathcal{C}_j(\Lambda_j) & \mathcal{D}_j(\Lambda_j) & \tilde{D}_{ju}^{(n_x)} \end{array}\right]' \begin{bmatrix} \Phi\otimes P_j & 0 & 0 \\ 0 & -\gamma_j I & 0 \\ 0 & 0 & I \end{bmatrix} \left[\begin{array}{cc|c} I & 0 & 0 \\ \mathcal{A}(\Lambda_j) & \mathcal{B}_j(\Lambda_j) & \tilde{B}_u^{(n_x)} \\ 0 & I & 0 \\ \mathcal{C}_j(\Lambda_j) & \mathcal{D}_j(\Lambda_j) & \tilde{D}_{ju}^{(n_x)} \end{array}\right]$$
$$+ \mathrm{He}\left\{ \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \begin{bmatrix} Z_j\tilde{C}_y^{(n_x)} & Z_j\tilde{D}_{yj}^{(n_x)} & -Y_j \end{bmatrix} \right\} \prec 0, \tag{23b}$$

where Z_j is given by

$$Z_j := U'\hat{\Theta} + Y_j\left(\begin{bmatrix} 0_{q\times q} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & 0_{n_u\times n_y} \end{bmatrix} - \Lambda_j \right),$$

then the reduced-order controller parameterized by

$$\Theta := \begin{bmatrix} Y_{11} & Y_{13} \\ Y_{31} & Y_{33} \end{bmatrix}^{-1} \hat{\Theta}V'$$

stabilizes the closed-loop system (7) *with a guaranteed upper bound on the closed-loop $\mathcal{H}_\infty$ performance $\|H_j\|_\infty^2 < \gamma_j$.*

The proof of Theorem 4 is analogous to the proof of Theorem 3, and is therefore omitted. For the sake of clarity, it is remarked that Φ_c or Φ_d should be substituted for Φ to arrive at the continuous, respectively, discrete-time synthesis conditions in Theorem 3 and Theorem 4. At the end of the section, we outline how the results of Theorem 3 and 4 can be adapted to the case where the initial controller (parameterized by Λ_j) is of reduced order.

Remark 3. *Note that the synthesis LMIs* (16a), (16b), *and* (23b) *contain the standard $\mathcal{H}_2$, respectively, $\mathcal{H}_\infty$ analysis LMIs for the performance of $H_j(\Lambda_j)$ as diagonal subblocks. This implies that*

$$\|H_j(\Lambda_j)\|_2^2 < \mu_j, \quad j \in \mathcal{S}_{\mathcal{H}_2}, \quad \|H_j(\Lambda_j)\|_\infty^2 < \gamma_j, \quad j \in \mathcal{S}_{\mathcal{H}_\infty}. \tag{24}$$

Remark 4. *Compared to the conditions that were presented in our earlier works [34, 35], the LMIs proposed in Theorem 3 and Theorem 4 are more general in the following ways. Firstly, the LMI conditions in this paper handle both continuous and discrete time, while the conditions in [34, 35] only apply to discrete-time dynamics. Secondly, the LMIs in this paper possess more degrees of freedom, which is clarified by the following particular choices of LMI variables from which the LMIs in [34, 35] are recovered: Select $A_{22} = 0$, and note that $0 \in \Omega(\Phi_\mathrm{d})$. Moreover, define an LMI variable $\tilde{\Theta} \in \mathbb{R}^{(q+n_u)\times(q+n_y)}$ such that $\hat{\Theta} := \tilde{\Theta}V$, substitute the latter in the LMI conditions* (16a), (16b) *and* (23b), *and note that $\tilde{\Theta}$ contains fewer scalar variables than $\hat{\Theta}$.*

By combining Theorem 3 and 4, the following convex optimization problem for reduced-order multi-objective controller synthesis is considered:

$$\begin{array}{lll} \text{minimize} & \sum_{j\in\mathcal{S}_{\mathcal{H}_2}} a_j\mu_j + \sum_{j\in\mathcal{S}_{\mathcal{H}_\infty}} a_j\gamma_j & \\ \text{subject to} & \operatorname{trace}(W_j) < \mu_j, \text{ LMI (16a), LMI (16b)}, & j \in \mathcal{S}_{\mathcal{H}_2} \\ & \text{LMI (23a), LMI (23b)}, & j \in \mathcal{S}_{\mathcal{H}_\infty} \\ & \mu_j < b_j, & j \in \mathcal{S}_{\mathcal{H}_2} \\ & \gamma_j < b_j, & j \in \mathcal{S}_{\mathcal{H}_\infty} \end{array} \tag{25}$$

where $a_j \geq 0$ and $b_j > 0$ for $j \in \mathcal{S}$ should be provided by the user. Note that b_j is a bound on $\|H_j\|_2^2$ for $j \in \mathcal{S}_{\mathcal{H}_2}$ and on $\|H_j\|_\infty^2$ for $j \in \mathcal{S}_{\mathcal{H}_\infty}$. The corresponding optimization variables are $P_j \in \mathbb{S}^{2n_x}$, $Y_j \in \mathbb{R}^{(n_x+n_u)\times(n_x+n_u)}$ for $j \in \mathcal{S}$, $\mu_j \in \mathbb{R}$, $W_j \in \mathbb{S}^{n_{zj}}$ for $j \in \mathcal{S}_{\mathcal{H}_2}$, $\gamma_j \in \mathbb{R}$ for $j \in \mathcal{S}_{\mathcal{H}_\infty}$ and $\hat{\Theta} \in \mathbb{R}^{(q+n_u)\times(n_x+n_y)}$. The controller order is characterized by the dimensions of the design variable $\hat{\Theta}$ and the structure of Y_j, $j \in \mathcal{S}$, as specified in Theorem 3 and Theorem 4. Note that two subblocks of Y_j, $j \in \mathcal{S}$ are set to zero, which is necessary to arrive at LMIs through the nonlinear change of controller variables (17). Furthermore, the matrices Y_{11}, Y_{13}, Y_{31} and Y_{33} are independent of $j \in \mathcal{S}$, since these matrices are used in the reconstruction of a single controller parameter Θ. This results in additional conservatism with respect to single-objective reduced-order synthesis. However, note the beneficial fact that the variables P_j, W_j, Y_{12j}, Y_{22j} and Y_{23j} can be chosen different for each $j \in \mathcal{S}$, and that moreover a different full-order controller parameter is allowed for each performance channel in the LMI conditions (16a), (16b) and (23b). These degrees of freedom are essential to reduce conservatism in the reduced-order multi-objective controller design.

Remark 3 implies that selecting a single controller parameter Λ (i.e. $\Lambda_j := \Lambda$, $j \in \mathcal{S}$), designed according to e.g. Lyapunov shaping or G shaping, results in an objective value of (25) that is always larger (not necessarily strictly) compared to the performance of $H(\Lambda)$. In addition, it is straightforward to show that substitution of $q = n_x$ in (25), a corresponding solution is characterized by $\Theta = \Lambda$ (if feasible upper bounds b_j, $j \in \mathcal{S}$ are selected). Based on these observations, it is advised to select different full-order controller parameters Λ_j, $j \in \mathcal{S}$ for reduced-order controller design. The benefits of this degree of freedom will be demonstrated in the next section.

Two extensions of the reduced-order synthesis LMIs are discussed now, demonstrating their flexibility and generality.

Strictly proper reduced-order controller design. In order to design strictly proper reduced-order controllers (i.e. $D_c = 0$), it is necessary to impose structural constraints on some of the LMI variables, which is due to the nonlinear transformation of controller variables (17). Subdividing the LMI variable $\hat{\Theta}$ as in (22), and imposing the constraints $\hat{\Theta}_{22} = 0$ and $Y_{31} = 0$, a strictly proper reduced-order controller results whenever

(25) is feasible. To see this, notice that (17) yields $\hat{\Theta}_{22} = Y_{31}B_c + Y_{33}D_c$. Consequently, the above constraints imply $Y_{33}D_c = 0$. Finally, positive definiteness of $Y_{33} + Y_{33}'$ gives $D_c = 0$ whenever (25) is feasible.

Start from a set of reduced-order controller parameters. Theorem 3 and Theorem 4 can be generalized in the following way. Given a set of fixed-order controller parameters $\{\Lambda_j \mid j \in \mathcal{S}\}$ corresponding to controllers of order p, where $0 \leq p \leq n_x$, a controller of order q $(0 \leq q < p)$ can be designed. The dimensions of the LMI parameters/variables generalize as follows:

$$U = \begin{bmatrix} I_q & 0_{q\times(p-q)} & 0 \\ 0 & 0 & I_{n_u} \end{bmatrix}, \quad V = \begin{bmatrix} I_q & 0_{q\times(p-q)} & 0 \\ 0 & 0 & I_{n_y} \end{bmatrix},$$

$P_j \in \mathbb{S}^{n_x+p}$, $\hat{\Theta} \in \mathbb{R}^{(q+n_u)\times(p+n_y)}$, $Y_{12j} \in \mathbb{R}^{q\times(p-q)}$, $Y_{22j} \in \mathbb{R}^{(p-q)\times(p-q)}$, $Y_{32j} \in \mathbb{R}^{n_u\times(p-q)}$. Moreover, the matrices $\tilde{B}_u^{(n_x)}$, $\tilde{C}_y^{(n_x)}$, $\tilde{D}_{yj}^{(n_x)}$, $\tilde{D}_{ju}^{(n_x)}$ should be replaced by $\tilde{B}_u^{(p)}$, $\tilde{C}_y^{(p)}$, $\tilde{D}_{yj}^{(p)}$, $\tilde{D}_{ju}^{(p)}$, respectively. This generalization allows iterative application of Theorem 3 and Theorem 4.

4. Numerical validation

To demonstrate the effectiveness of the presented LMI approach, it is applied to three numerical examples and compared with existing controller design approaches. The LMIs are implemented and solved in MATLAB, using the software packages Yalmip [38] and SeDuMi [39]. Exact $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms are calculated a posteriori with `norm.m`.

First, a single-objective reduced-order $\mathcal{H}_\infty$ control problem is considered in Subsection 4.1. Then, the design of reduced-order multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ controllers is considered in Subsections 4.2 and 4.3, for a continous-time, respectively, discrete-time system. The latter two examples demonstrate that the freedom to select different full-order controllers for the different performance channels can be favorably exploited to obtain a significant reduction of conservatism.

4.1. Example 1: reduced-order $\mathcal{H}_\infty$ control

This example considers the $AC7$ aircraft model from the COMPl$_e$ib library [40, 41]. This 9$^{\text{th}}$ order LTI model is discretized using zero-order hold at a sampling period of $0.01s$. The objective is to design reduced-order controllers for the discretized LTI system, optimizing the $\mathcal{H}_\infty$ performance on the channel $w \to z$.

First a full-order optimal $\mathcal{H}_\infty$ controller is computed, yielding a closed-loop $\mathcal{H}_\infty$ norm of $4.0 \cdot 10^{-2}$. Using this full-order $\mathcal{H}_\infty$ controller as parameter, selecting $A_{22} = 0$ and minimizing γ subject to the LMIs (23a) and (23b), controllers of order $q = 2, \ldots, 8$ are obtained. For $q = 0, 1$, the LMI problem is infeasible when the optimal full-order $\mathcal{H}_\infty$ controller parameter is used. However, a feasible solution might be obtained by selecting a suboptimal full-order controller parameter. Therefore, a suboptimal controller is computed by fixing the upper bound γ in the full-order synthesis LMI to a feasible value of $\gamma = (0.3)^2$. Solving the corresponding feasibility problem with SeDuMi results in a closed-loop $\mathcal{H}_\infty$ norm of $6.4 \cdot 10^{-2}$. Subsequently, this controller is used as a parameter in the reduced-order $\mathcal{H}_\infty$ synthesis LMIs, yielding feasible solutions for $q = 0, 1$. Table 1 summarizes the results. The controller order is given in the first row, while the performance upper bounds $\sqrt{\gamma}$ and the closed-loop $\mathcal{H}_\infty$ norms are given in the second row. Note that optimal $\mathcal{H}_\infty$ controllers are obtained for high orders $(q \geq 6)$, since they have the same closed-loop performance as the optimal full-order $\mathcal{H}_\infty$ controller, see also Remark 3.

Using the generalization that is discussed in Subsection 3.2, reduced-order controllers are computed in an iterative fashion. Specifically, starting from a full-order controller, controllers of order q are computed by using the controller of order $q+1$ as a parameter for $q = n_x - 1, \ldots, 0$. The corresponding results are given in row 3 of Table 1. It is remarkable that an optimal controller of order 5 is computed, and that the upper bound corresponding to the design of a static output feedback controller significantly improved compared to the result in the second row.

Table 1: A comparison between (iterative) application of Theorem 4 and the approaches [27, 42, 43] confirms the potential of the approach of this paper. For each approach and controller order q, the upper bound and closed-loop $\mathcal{H}_\infty$ performance are indicated by $\sqrt{\gamma}$, respectively, $\|H\|_\infty$.

row 1		q	8	7	6	5	4	3	2	1	0
row 2	Th. 2	$10^2 \cdot \sqrt{\gamma}$	4.0	4.1	4.1	5.0	5.1	5.0	11.1	8.0	43.0
		$10^2 \cdot \|H\|_\infty$	4.0	4.0	4.0	5.0	5.0	4.7	5.9	6.8	6.8
row 3	Th. 2 iter.	$10^2 \cdot \sqrt{\gamma}$	4.0	4.1	4.1	4.1	5.0	5.5	10.4	8.7	9.6
		$10^2 \cdot \|H\|_\infty$	4.0	4.0	4.0	4.0	5.0	5.3	6.8	6.4	6.9
row 4	[27]	$10^2 \cdot \sqrt{\gamma}$	5.8	6.5	7.8	11.5	11.9	13.4	14.0	15.4	10.2
		$10^2 \cdot \|H\|_\infty$	4.3	4.7	6.5	6.5	6.5	7.4	6.7	7.1	6.6
row 5	[42, 43]	$10^2 \cdot \|H\|_\infty$	4.0	4.0	4.0	4.0	4.0	5.2	4.0	6.5	6.5

Table 2: A comparison between computation times (in seconds) corresponding to the LMIs of Theorem 4 and HIFOO [42, 43] demonstrates that the approach of this paper is numerically more attractive.

q	8	7	6	5	4	3	2	1	0
Th. 2	1.00	0.72	0.69	0.78	0.75	0.63	0.39	0.48	0.36
Th. 2 iter.	0.99	0.75	0.46	0.32	0.34	0.26	0.15	0.17	0.14
[42, 43]	102	73.0	89.1	59.6	47.2	65.9	43.3	1.56	0.72

The discrete-time counterpart of the approach discussed in [27], see also [32, 33], is applied now. This approach relies on the computation of a state feedback controller for an augmented system, which is subsequently used as a parameter in a sufficient LMI for reduced-order $\mathcal{H}_\infty$ controller design. It is worth remarking that a different state feedback controller is required for the computation of controllers of different orders q, whereas the approach discussed in this paper allows the use of a single full-order controller to compute reduced-order controllers of different orders. Starting from an optimal $\mathcal{H}_\infty$ state feedback, the approach from [27] returns infeasibility for all orders $q = 0, \ldots, 8$. Therefore, the upper bound $\sqrt{\gamma} = 4.0 \cdot 10^{-2}$, corresponding to the performance of an optimal full-order controller, is fixed in the LMI for state feedback design, resulting in suboptimal state feedback controllers. The latter controllers are subsequently used for reduced-order controller design. The results are shown in row 4 of Table 1. A comparison reveals that, except for $q = 0$, the approach of this paper yields controllers with better $\mathcal{H}_\infty$ performance.

The last row of Table 1 summarizes the results of the HIFOO package [42, 43]. The continuous-time $AC7$ model is used, since HIFOO requires a continuous-time model. Two randomly generated starting points are used, and the option to restrict to the BFGS phase is selected. While HIFOO yields lower $\mathcal{H}_\infty$ performances in some cases, the approach in this paper provides better results than HIFOO for $q = 3$, and similar results are obtained for $q = 1$ and $q = 5, \ldots, 8$. It it worth mentioning that the LMI approach is considerably faster than HIFOO, as Table 2 indicates. For each order q specified in the first row, the second row indicates the computation times that correspond to solving the LMIs from Theorem 4, starting from a full-order $\mathcal{H}_\infty$ controller. Note that the time to compute a full-order controller, which is $0.55s$ for the optimal $\mathcal{H}_\infty$ controller, should be added to obtain the total computation time. The third row shows the computation times resulting from iterative application of Theorem 4, which are lower for lower controller orders due to fewer LMI rows and variables. In this case, the total computation time follows by adding the computation times of all higher order controllers. It is clear from the last row that, compared to the aforementioned approaches, the computation times of HIFOO are similar for low order controller design, but substantially higher for computation of high order controllers.

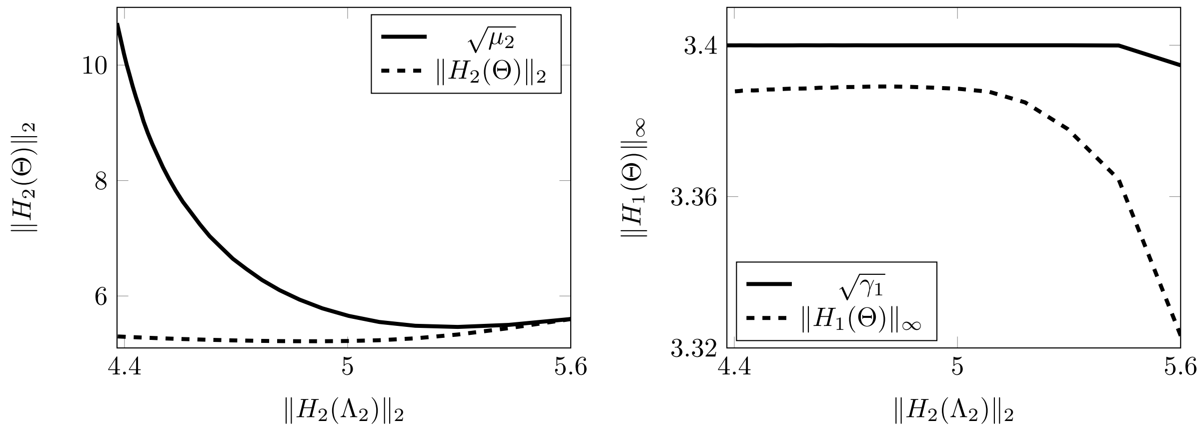

Figure 1: Using different Lyapunov shaping controllers Λ_j, $j \in \mathcal{S}$, allows the design of reduced-order controllers which are less conservative than a Lyapunov shaping controller Λ.

4.2. Example 2: reduced-order $\mathcal{H}_2/\mathcal{H}_\infty$ control, continuous time

Consider the $AC3$ model from the COMPl$_e$ib library [40, 41]. The objective is to design a reduced-order multi-objective controller for this 5^{th} order aircraft model. Denoting the exogenous input and the regulated output of the $AC3$ model by w and z, respectively, the following performance channels are considered: $w_1 = w_2 = w$ and $z_1 = z_2 = z$, with $\mathcal{S}_{\mathcal{H}_2} = \{2\}$ and $\mathcal{S}_{\mathcal{H}_\infty} = \{1\}$. The $\mathcal{H}_2$ performance is minimized subject to a bound on the $\mathcal{H}_\infty$ performance.

First, the Lyapunov shaping paradigm is applied to design a full-order controller Λ, using the $\mathcal{H}_\infty$ performance bound $\|H_1(\Lambda)\|_\infty^2 < b_{LS} = (3.4)^2$. The obtained controller yields a closed-loop $\mathcal{H}_2$ performance $\|H_2(\Lambda)\|_2 = 5.60$. Subsequently, $\Lambda_1 = \Lambda_2 = \Lambda$ is selected in the LMIs to design a reduced-order controller Θ of order 1, by solving the convex optimization problem (25) with $q = 1$, $A_{22} = -I$, objective function μ_2 and bounds $b_1 = (3.4)^2$ and $b_2 = \infty$. The solution is promising, since the resulting upper bound is tight and similar performance compared to the full-order controller is obtained: $\sqrt{\mu_2} = 5.60$ and $\|H_2(\Theta)\|_2 = 5.60$.

However, potentially less conservative reduced-order controllers can be computed by exploiting the freedom of using two different full-order controller parameters Λ_j, $j = 1, 2$. Therefore, the Lyapunov shaping design $\Lambda_1 = \Lambda$ is substituted in the LMI (23b) to guarantee the constraint $\gamma_1 < b_1$, and a second full-order controller Λ_2 is designed by modifying the bound b_{LS} in the Lyapunov shaping design step. Selecting a value $b_{LS} > b_1$ yields better closed-loop $\mathcal{H}_2$ performance compared to Λ_1: $\|H_2(\Lambda_2)\|_2 < \|H_2(\Lambda_1)\|_2$. As a consequence, substitution of Λ_2 in the $\mathcal{H}_2$ synthesis LMIs (16a) and (16b) allows a lower $\mathcal{H}_2$ performance upper bound in the reduced-order controller design step.

The left part of Figure 1 shows the $\mathcal{H}_2$ upper bound $\sqrt{\mu_2}$ and $\mathcal{H}_2$ performance $\|H_2(\Theta)\|_2$ for different values of $\|H_2(\Lambda_2)\|_2$. The rightmost point corresponds to the choice $\Lambda_2 = \Lambda_1 = \Lambda$. Gradually improving the closed-loop $\mathcal{H}_2$ performance corresponding to Λ_2 completes the left part of Figure 1, revealing that using a controller Λ_2 resulting in a closed-loop performance $\|H_2(\Lambda_2)\|_2 \approx 4.9$ is optimal in this case: $\|H_2(\Theta)\|_2 = 5.22$. This corresponds to an improvement of almost 7% in performance compared to the full-order Lyapunov shaping design. Selecting, for example, $\|H_2(\Lambda_2)\|_2 = 5.2$ results in an improvement both in terms of the upper bound and the closed-loop $\mathcal{H}_2$ performance: $\sqrt{\mu_2} = 5.48$ and $\|H_2(\Theta)\|_2 = 5.27$. As can be inferred from the right part of Figure 1, the improvement in $\mathcal{H}_2$ performance results in a slight increase of the closed-loop $\mathcal{H}_\infty$ norm.

4.3. Example 3: reduced-order $\mathcal{H}_2/\mathcal{H}_\infty$ control, discrete time

A multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ controller design example is considered, see Example 4 in [18]. The following LTI system is studied

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0.5 & 0 \\ 0 & 1 & -0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} w(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k), \\ z(k) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(k), \\ y(k) &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} w(k). \end{aligned} \tag{26}$$

Four performance channels are considered, hence $\mathcal{S} = \{1, 2, 3, 4\}$, with $\mathcal{S}_{\mathcal{H}_\infty} = \{1, 2, 3\}$ and $\mathcal{S}_{\mathcal{H}_2} = \{4\}$. The corresponding selection matrices are given by

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}', \quad R_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}', \quad R_3 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}', \quad R_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}',$$

and

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \quad L_4 = I_4.$$

We are interested in the computation of a reduced-order dynamic output feedback controller Θ, such that

$$\|H_j(\Theta)\|_\infty^2 < b, \quad j \in \mathcal{S}_{\mathcal{H}_\infty} \tag{27}$$

for some predefined fixed value of $b > 0$, and moreover the closed-loop $\mathcal{H}_2$ performance $\|H_4(\Theta)\|_2$ is minimized. A bound $b = (7.4)^2$ is selected.

First a full-order controller Λ is computed using the G shaping paradigm [18], where an upper bound on $\|H_4(\Lambda)\|_2^2$ is minimized subject to the constraints $\|H_j(\Lambda)\|_\infty^2 < b_{GS} = (7.4)^2$. This results in a closed-loop $\mathcal{H}_2$ norm of $\|H_4(\Lambda)\|_2 = 16.11$.

For the design of a reduced-order controller ($q = 2$), the matrix A_{22} is set to zero. To guarantee (27), it seems intuitive to select $\Lambda_j = \Lambda$, $j = 1, 2, 3$. When $\Lambda_4 = \Lambda$ is also selected, solving the convex optimization problem (25) with objective function μ_4 and $b = (7.4)^2$ yields $\sqrt{\mu_4} = 16.11$ and $\|H_4(\Theta)\|_2 = 15.63$. The upper bound $\sqrt{\mu_4}$ is equal to $\|H_4(\Lambda)\|_2$, which is theoretically the best achievable according to (24). However, note that the closed-loop $\mathcal{H}_2$ performance corresponding to the reduced-order controller Θ is better compared to the G shaping controller Λ.

Now a different controller parameter Λ_4 is designed using the G shaping paradigm, by imposing a a bound $b_{GS} > (7.4)^2$ on the $\mathcal{H}_\infty$ performances. In this way the $\mathcal{H}_\infty$ constraints are relaxed, resulting in $\|H_4(\Lambda_4)\|_2 < \|H_4(\Lambda)\|_2$. In turn, this allows a better $\mathcal{H}_2$ performance corresponding to the solution of (25) when Λ_4 is substituted in the $\mathcal{H}_2$ synthesis LMIs.

Figure 2 shows the upper bound $\sqrt{\mu_4}$ and $\mathcal{H}_2$ performance $\|H_4(\Theta)\|_2$ (left), and the maximum of the upper bounds γ_j and $\mathcal{H}_\infty$ performances $\|H_j(\Theta)\|_\infty$, $j = 1, 2, 3$ (right), corresponding to the reduced-order design as a function of $\|H_4(\Lambda_4)\|_2$. The rightmost point of the subfigures corresponds to the choice $\Lambda_4 = \Lambda$, and the figure is completed by gradually improving the $\mathcal{H}_2$ performance corresponding to Λ_4. The left part of Figure 2 implies that selecting a controller parameter Λ_4 with a slightly better closed-loop $\mathcal{H}_2$ performance $\|H_4(\Lambda_4)\|_2$ than Λ results in a lower upper bound μ_4 and better $\mathcal{H}_2$ performance. For example, taking $b_{GS} = (7.9)^2$ yields $\|H_4(\Lambda_4)\|_2 = 15.4$, resulting in $\sqrt{\mu_4} = 15.6$ and $\|H_4(\Theta)\|_2 = 14.9$, demonstrating an improvement of 7.5% in $\mathcal{H}_2$ performance compared to the G shaping controller Λ. Selecting $b_{GS} > (8.5)^2$ in the G shaping design step, no feasible solution was obtained. Looking at the right part of Figure 2, the $\mathcal{H}_2$ performance improves at the expense of a higher $\mathcal{H}_\infty$ norm, approximating the $\mathcal{H}_\infty$ bound for lower values of $\|H_4(\Lambda_4)\|_2$.

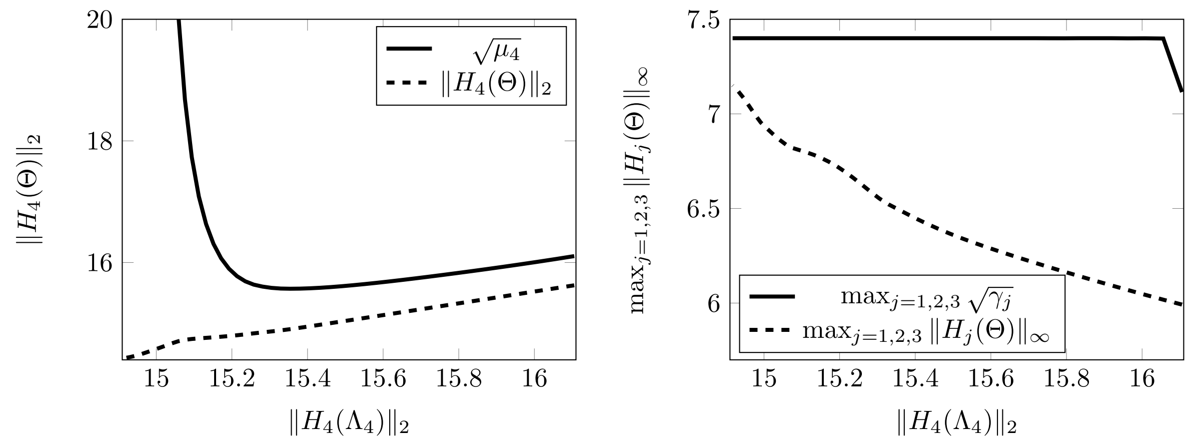

Figure 2: Using different G shaping controllers Λ_j, $j \in \mathcal{S}$, allows the design of reduced-order controllers which are less conservative than a G shaping controller Λ.

Remark 5. *Although, theoretically speaking, the choice of the A_{22} matrix might influence the performance realized by the reduced-order controller, extensive numerical experiments have shown that selecting A_{22} different from the values suggested in the paper ($A_{22} = 0$ in discrete-time and $A_{22} = -I$ in continuous-time) has a negligible effect on the closed-loop performance.*

5. Conclusions

A novel projection lemma based LMI framework for the design of reduced-order multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ controllers for LTI systems is presented. In this framework, continuous time and discrete time are unified. The starting point is a set of full-order $\mathcal{H}_2/\mathcal{H}_\infty$ controllers, which are used as parameters in sufficient LMIs for reduced-order $\mathcal{H}_2/\mathcal{H}_\infty$ controller design.

Our previous results [34, 35] have been generalized, due to extensions to continuous-time problems, the addition of extra degrees of freedom in the LMIs, and a reduction of the number of LMI rows. Furthermore, two extensions of the LMI framework are proposed, demonstrating their flexibility and generality. Firstly, the reduced-order synthesis LMIs allow the design of strictly proper reduced-order controllers. Secondly, it is shown how to extend the LMIs to start from a given set of reduced-order controller parameters.

The merits of our LMI approach are illustrated by means of numerical comparisons with existing approaches, both for single- and multi-objective reduced-order synthesis problems. The examples in the Subsections 4.2 and 4.3 demonstrate that our LMI approach enables the computation of reduced-order controllers which are potentially less conservative than full-order designs resulting from the Lyapunov and G shaping paradigm. The latter is a result of allowing a different full-order controller parameter in the LMIs corresponding to each performance specification.

Extensions of the LMI framework to reduced-order linear parameter-varying controller design are planned. Furthermore, applying the approach for full-order and reduced-order robust controller design for uncertain linear systems is future work.

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